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Multiple minima in glassy random-matrix models

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Abstract. Certain simple models of structural glasses (Cugliandolo L F, Kurchan J, Parisi G and Ritort F 1995 *Phys. Rev. Lett.* **74** 1012, Parisi G 1997 Statistical properties of random matrices and the replica method *Preprint* cond-mat/9701032) map onto random-matrix models. These random-matrix models have gaps in their eigenvalue distributions. It turns out that matrix models with gaps in their eigenvalue distributions have the unusual property of multiple solutions or minima of the free energy at the same point in phase space. I present evidence for the presence of multiple solutions in these models both analytically and numerically. The multiple solutions have different free energies and observable correlation functions, the differences arising at higher order in $1/N$. The system can get trapped into different minima depending upon the path traversed in phase space to reach a particular point. The thermodynamic limit also depends upon the sequence by which N is taken to infinity (e.g. odd or even N), which is reminiscent of the structure discussed for another model for glasses (Marinari E, Parisi G and Ritort F 1994 *J. Phys. A: Math. Gen.* **27** 7615). Hence it would be of interest to study the landscape of these multiple solutions and determine whether it corresponds to a supercooled liquid or glass.

1. Introduction

There has been a close connection between developments in glasses (for example, spin glasses) and random-matrix models. In particular in [1,2], it has been shown that certain simple models of structural glasses map onto certain random-matrix models in the high-temperature region. These matrix models have gaps in their densities of eigenvalues. Thus a detailed study of these random-matrix models with gaps in their eigenvalue spectra is of relevance to structural glasses. I find that these matrix models have the unusual and new property of multiple solutions at the same point in phase space. I provide evidence both analytically and numerically for the existence of these multiple solutions. I then describe some properties of the multiple solutions that differ from each other in their free energies, correlation functions and eigenvalue distributions at higher orders in $1/N$ where N is the size of the matrix. The structure of the landscape of these multiple solutions remains an open question, a study of which would help identify correspondences with glass or supercooled liquid behaviour.

In section 2 the connection between a simple model of structural glasses and random-matrix models is described. It is emphasized that the latter have gaps in their eigenvalue spectra. In section 3, I demonstrate that such random-matrix models have multiple solutions and describe the properties of these solutions. Finally, section 4 contains conclusions and open questions.

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2. The connection of a simple model of structural glasses with random-matrix models

We begin by building up a connection between random-matrix models and the study of structural glasses in the high-temperature phase based primarily on the work of reference [1,2]. The Hamiltonian

$$H = \frac{1}{n}\sum_{a\neq b}(\vec{s}^a \cdot \vec{s}^b)^p$$

corresponds to n particles moving in an N-dimensional space, in the limit $n, N \to \infty$. The coordinates of the particles are $\vec{s}^a = (s_1^a, \ldots, s_N^a)$ where $a = 1, \ldots, n$; the parameter p is a real number. In the 'spherical' case the particles are constrained to move on the surface $|s^a|^2 = N$, for all a. In the $\pm$ case they occupy only the vertices of a hypercube $s_i^a = \pm 1$. The case of $p = 2$ is studied.

The corresponding partition function is

$$Z = \mathrm{e}^{\beta N^2}\,\mathrm{Tr}_{\{s\}}\exp\left[-\frac{\beta}{\alpha N}\,\mathrm{tr}(S^\dagger S S^\dagger S)\right]$$

where S is the $N \times n$ rectangular matrix of elements s_i^a, and $\mathrm{Tr}_{\{s\}}$ runs over either the spherical or the Ising measure (at high temperature the spherical and Ising case have the same partition function; this is shown in reference [1]). Here $n = \alpha N$ with $\alpha \geqslant 1$. Using the large-N equivalence with the global constraint we get for the spherical model

$$Z_{sph} \approx \mathrm{e}^{\beta N^2}\int \mathrm{d}\mu \int \prod_{i=1}^{N}\mathrm{d}x_i\,\exp\left(-N^2\frac{\beta\mu}{2} - \beta E[x]\right).$$

The $x_i\sqrt{N}$ are the 'diagonal' values of S in its canonical form and $E[x]$ where $x = \{x_i\}$ gives

$$E[x] = N\sum_{i=1}^{N}\left(\frac{1}{\alpha}x_i^4 - \frac{\mu}{2\alpha}x_i^2 - \frac{(\alpha-1)}{\beta}\ln|x_i|\right) - \frac{1}{2\beta}\sum_{i\neq j}\ln|x_i^2 - x_j^2|. \tag{2.1}$$

This 'energy' functional has a log term for the repulsion between the eigenvalues and a potential

$$V(x_i) = \frac{1}{\alpha}x_i^4 - \frac{\mu}{2\alpha}x_i^2 - \frac{(\alpha-1)}{\beta}\ln|x_i|$$

seen by each eigenvalue, analogous to the case for a Dyson gas. It is evident that $V(x)$ has a double-well structure for values of parameters relevant for the glass model. A consequence of this and the repulsion is that in the large-N limit, the eigenvalues x_i reside in two bands separated by a gap. In this case the spectrum also has Z_2 symmetry ($x_i \to -x_i$). The matrix model is a form of the generalized Penner model or the Penner–Kontsevich model.

These matrix models have multiple solutions at the same point in phase space, characterized by different free energies and correlation functions. Think of the matrix model as occupying a point in the phase space of coupling constants, which is higher dimensional (for example, think of $V(x_i)$ as having additional terms $\sigma_1 x_i$, $\sigma_3 x_i^3$, ...). Let us analyse the neighbouring points in phase space and approach this point by taking the limit $\sigma_k \to 0$ along some path. Then one gets different solutions for the free energy and correlation functions, depending upon the path chosen. This might mirror the path dependence of a structural glass as it is cooled, in the same spirit as the universality of the Wigner–Dyson statistics for a random-matrix model with respect to the phase-space point mirrors the observed universality of eigenvalue statistics in real chaotic systems. In the large-N limit there are an infinite number of solutions whose free energies and correlation functions differ from each other by terms of

order $1/N$. It would be interesting to explore the physical meaning of the different solutions in terms of the glass energy landscape and their consequence for the glass transition. This exploration is left for the future. Here I simply demonstrate the existence of these solutions.

3. Multiple solutions in matrix models with gaps

Let M be an $N \times N$ Hermitian matrix. The partition function to be considered is

$$Z = \int \mathrm{d}M \, \mathrm{e}^{-N \operatorname{tr} V(M)}.$$

The Haar measure is

$$\mathrm{d}M = \prod_{i=1}^{N} \mathrm{d}M_{ii} \prod_{i<j} \mathrm{d}M_{ij}^{(1)} \, \mathrm{d}M_{ij}^{(2)} \qquad \text{with } M_{ij} = M_{ij}^{(1)} + \mathrm{i}M_{ij}^{(2)}.$$

The potential for a general Penner matrix model is $V(M) = V_0(M) - t \log M$, where $V_0(M)$ is a polynomial and t a parameter. If $V_0(M) = \frac{1}{2}\mu M^2$ the model is a Gaussian Penner model where we can rewrite $\log M = \frac{1}{2} \log M^2$. This has Z_2 symmetry and the potential is a double well with eigenvalues distributed in disconnected segments. The partition function for the Gaussian Penner model in terms of its eigenvalues x_i is

$$Z = \int \prod \mathrm{d}x_i \, \exp[-E(x)] \tag{3.1}$$

where

$$E(x) = N \sum_{i=1}^{N} \left(\frac{\mu}{2} x_i^2 - \frac{t}{2} \log x_i^2 \right) - 2 \sum_{i \neq j} \log |x_i - x_j|.$$

This is very similar to the structure of equation (2.1), i.e. a repulsion term and a double-well potential involving $\log |x|$. I will show explicitly that this has multiple solutions.

It is convenient for pedagogical reasons to consider a model with a polynomial potential:

$$V(M) = g_1 M + (g_2/2)M^2 + (g_3/3)M^3 + (g_4/4)M^4 + \cdots.$$

In particular, for the symmetric M^4-matrix model, $g_2 = \mu$, $g_4 = g$ and $g_1, g_3, g_5, \ldots = 0$. The spectrum or the density of eigenvalues

$$\rho(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - \lambda_i)$$

is in the large-N limit or doing the saddle-point analysis just the Wigner semi-circle for a quadratic potential (Gaussian probability distribution for the eigenvalues). The simplest way to determine $\rho(x)$ explicitly is to use the generating function

$$F(z) = \left\langle \frac{1}{N} \operatorname{Tr} \frac{1}{z - M} \right\rangle.$$

The density $\rho(x)$ is then determined by the formula

$$\rho(x) = -\frac{1}{2\mathrm{i}\pi} (F(x + \mathrm{i}0) - F(x - \mathrm{i}0)).$$

The partition function Z can be rewritten in terms of the orthogonal polynomials P_n where the polynomials are defined as

$$\int_{-\infty}^{\infty} \mathrm{d}\lambda \, \mathrm{e}^{-NV(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{nm}$$

and normalized such that $P_n(x) = x^n + c_1 x^{(n-1)} + \cdots$. Then $Z = N! h_0 h_1 h_2 \cdots h_{N-1}$. The P_n satisfy recurrence relations [4]:

$$x P_n = P_{n+1} + S_n P_n + R_n P_{n-1} \tag{3.2}$$

which defines the 'recurrence coefficients' R_n and S_n. From the P_n one can find the R_n and S_n and vice versa. From the definition, it follows that recurrence coefficients are given by certain integrals, e.g.

$$S_0 = \frac{I_1}{I_0} \qquad R_0 = 0 \qquad R_1 = \frac{I_2}{I_0} - \left(\frac{I_1}{I_0}\right)^2$$

where

$$I_n = \int \mathrm{d}x\; x^n \mathrm{e}^{-NV(x)}$$

and so on. In general the free energies and correlation functions can be expressed in terms of the recurrence coefficients R_n and S_n. A set of recurrence coefficients corresponds to a solution of the model. In particular, it follows that $h_n = h_{n-1} R_n$; hence the free energy is

$$\Gamma = \ln Z = \ln N! + N \ln h_0 + \sum_{n=1}^{N-1} (N-n) \ln R_n.$$

The problem then reduces to finding the recurrence coefficients R_n and S_n.

The recurrence coefficients satisfy recurrence relations that follow from the identities [4]

$$\text{I:} \qquad \int \mathrm{d}x\; \mathrm{e}^{-NV(x)} P_n(x) V'(x) P_n(x) = 0 \tag{3.3}$$

and

$$\text{II:} \qquad n h_{n-1} = N \int \mathrm{d}x\; \mathrm{e}^{-NV(x)} P_n(x) V'(x) P_{n-1}(x). \tag{3.4}$$

For example for

$$V(x) = \frac{\mu}{2} x^2 + \frac{g}{4} x^4$$

recurrence relation I is

$$0 = \mu S_n + g[R_{n+1}(S_{n+1} + 2S_n) + R_n(2S_n + S_{n-1}) + S_n^3]$$

while II is

$$\frac{n}{N} = \mu R_n + g R_n (R_{n-1} + R_n + R_{n+1}) + S_n^2 + S_{n-1}^2 + S_{n-1} S_n.$$

These identities together with boundary conditions ($S_{-1} = R_{-1} = 0$, S_0, $R_0 = 0$ and $R_1 = I_2/I_0$) in principle determine the recurrence coefficients uniquely. However, one discovers multiple solutions if one relaxes the boundary conditions, which is a physically reasonable thing to do when $V(x)$ has multiple wells or the spectrum has gaps, as will be discussed below. For Z_2-symmetric $V(x)$, $S_n = 0$ is always a solution (we refer to this as the symmetric solution, as it respects the Z_2 symmetry of the potential). However, there can be other solutions, depending upon the path chosen.

(1) *The one-band case (no gaps in the eigenvalue spectrum)*, e.g.

$$V(M) = \frac{\mu}{2} M^2 + \frac{g}{4} M^4 \qquad \text{with } \mu > 0, g > 0.$$

Then $S_n = 0$ from recurrence relation I. Recurrence relation II is

$$\frac{n}{N} = R_n[\mu + g(R_{n-1} + R_n + R_{n+1})]. \tag{3.5}$$

For the large-N limit we make the *ansatz* that R_n is a smooth function of $x \equiv n/N$, and expand as follows:

$$R_n = R(x) + \frac{1}{N}R_1(x) + \frac{1}{N^2}R_2(x) + \cdots.$$

Then equation (3.5) implies that $x = R(x)[\mu + 3gR(x)]$ with the solution

$$R(x) = \frac{1}{6g}[-\mu \pm \sqrt{\mu^2 + 12gx}].$$

This fits very well with the numerical evaluation of R_n which can be approximated by a smooth curve at large N as shown in figure 1.

(2a) *The two-band case (the eigenvalue spectrum has a gap)*, e.g.

$$V(M) = \frac{\mu}{2}M^2 + \frac{g}{4}M^4 \qquad \text{with } \mu < 0,\ g > 0.$$

Consider again equation (3.5) with $n/N \to x$ and $S_n = 0$. However, the assumption of a single smooth function describing R_n is not correct. Figure 2 shows the numerical result for R_n which suggests that two smooth functions are needed to describe R_n (a 'period-2' *ansatz*). The following *ansatz* works:

$$R_n = \begin{cases} A_n = A(x) + \dfrac{1}{N}A_1(x) + \cdots & \text{for } n \text{ even until } x = \bar{x} \\ B_n = B(x) + \dfrac{1}{N}B_1(x) + \cdots & \text{for } n \text{ odd} \end{cases} \tag{3.6}$$
$$R_n = R(x) + \frac{1}{N}R_1(x) + \cdots \qquad \text{for all } n,\ n > \bar{n} = N\bar{x}.$$

Substituting equation (3.6) into equation (3.5) and equating equal powers of $1/N$ we get, for $x < \bar{x} = \mu^2/(4g)$,

$$\begin{aligned} A(x) &= \frac{1}{2g}[|\mu| - \sqrt{\mu^2 - 4gx}] \\ B(x) &= \frac{1}{2g}[|\mu| + \sqrt{\mu^2 - 4gx}] \end{aligned} \tag{3.7}$$

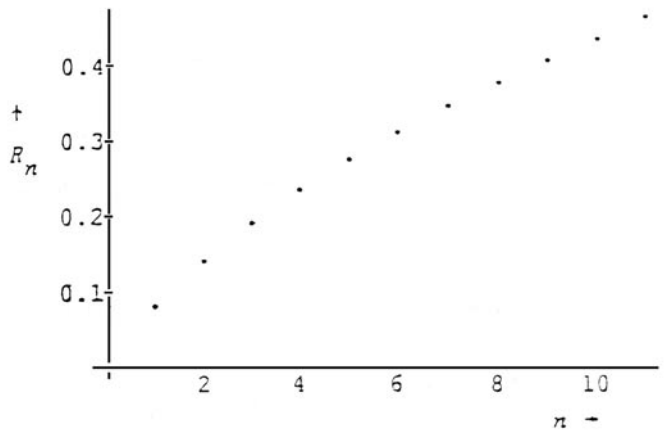

Figure 1. Numerical results for R_n for a single well with $N = 10$, $\mu = 1$ and $g = 1$.

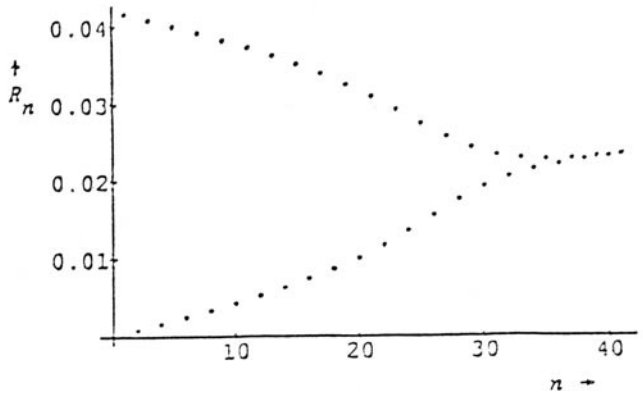

Figure 2. Numerical results for R_n for a double well with $N = 25$, $\mu = -105$ and $g = 2500$. R_n for odd and even n fall on different 'smooth curves'.

and for $x \geqslant \bar{x}$

$$R(x) = \frac{1}{6g}[-\mu + \sqrt{\mu^2 + 12gx}]. \tag{3.8}$$

The above analytical result and numerical figure 2 agree very well.

(2b) *Multiple solutions.* The simplest way to understand the existence of multiple solutions which appear in this matrix model is to consider the integral

$$S_0 = \frac{1}{I_0} \int \mathrm{d}x \; x \mathrm{e}^{-NV(x)}.$$

We will approach the Z_2-symmetric point in phase space by taking $\sigma \to 0$ in the potential

$$V(x) = \sigma x + \frac{\mu}{2}x^2 + \frac{g}{4}x^4 \qquad \text{with } \mu < 0, g > 0.$$

The above solution for $R(x)$ is only one of an infinite family. The integral S_0 from $[-\infty, \infty]$ approaches zero if we take $\sigma = 0$ and then take $N \to \infty$, but if we take $\sigma \neq 0$ first, evaluate the integral, then take the limit $N \to \infty$ and afterwards $\sigma \to 0$, the integral is $\pm\sqrt{-\mu/g}$. This suggests to us that there may be multiple solutions in these models, as I_1 does not tend to the same limit when the limits of N going to infinity and the asymmetry parameter σ going to zero are interchanged.

A more precise way of establishing the presence of multiple solutions in these models is by using the recurrence coefficients. Let us relax the condition $S_n \neq 0$; we will find that this will be interesting. We use the period-2 *ansatz* for both R_n and S_n. Then

$$\begin{aligned} &R_n \to A_n \qquad S_n \to C_n \qquad \text{even } n \\ &R_n \to B_n \qquad S_n \to D_n \qquad \text{odd } n \end{aligned} \tag{3.9}$$

and I and II reduce to

$$\begin{aligned} &n \text{ even:} \qquad x = A[\mu + g(2B + A + C^2 + D^2 + CD)] \\ &n \text{ odd:} \qquad x = B[\mu + g(2A + B + C^2 + D^2 + CD)] \\ &n \text{ even:} \qquad 0 = \mu C + g(B(D + 2C) + A(D + 2C) + C^3) \\ &n \text{ odd:} \qquad 0 = \mu D + g(A(C + 2D) + B(C + 2D) + D^3). \end{aligned} \tag{3.10}$$

There are four equations and four unknowns here, but after some work we find that there are only three independent equations. The three independent equations are

$$\begin{aligned} C + D &= 0 \\ A + B + C^2 &= \frac{-\mu}{g} \\ AB &= \frac{x}{g}. \end{aligned} \tag{3.11}$$

Thus there is an infinite family of solutions corresponding to one functional degree of freedom. For example let us take the two extreme solutions:

(a) the 'symmetric solution': $C - D = 0$; then equation (3.11) implies $C = D$, and we recover equation (3.8) for $A(x)$ and $B(x)$;
(b) the 'maximally asymmetric solution': $A - B = 0$; then equation (3.11) implies

$$A = B = R = \sqrt{\frac{x}{g}} \qquad \text{and} \qquad C = -D = \left[\frac{|\mu|}{g} - \sqrt{\frac{4x}{g}}\right]^{1/2}.$$

Both have the same tree-level eigenvalue density and free energy, as the generating function is (see reference [5]) given by

$$F(z) = \int_0^1 \mathrm{d}x \, \frac{2z - (C + D)}{\sqrt{(z^2 - z(C + D) - (A + B - CD))^2 - 4AB}}. \tag{3.12}$$

This formula contains precisely the same three combinations as are fixed by the recurrence relation equation (3.11). Therefore, independently of which solution is chosen, we get the same F, and hence ρ and Γ at large N. This demonstrates the presence of multiple solutions from the recurrence coefficient point of view.

A numerical demonstration of the existence of multiple solutions proceeds as follows. The recurrence relations I and II follow by extremizing an effective potential with respect to R_n and S_n. For

$$V(x) = \sigma x + \frac{\mu}{2}x^2 + \frac{g}{4}x^4$$

this is given by

$$\begin{aligned} V_{eff} = \sum_{n=0}^{\infty} \Bigg\{ & \frac{-n}{N} \ln R_n + \mu R_n + \frac{g}{2}(R_n^2 + 2R_n R_{n+1}) + \sigma S_n + \frac{\mu}{2} S_n^2 \\ & + \frac{g}{4} S_n^4 + g R_n (S_n^2 + S_{n-1}^2 + S_{n-1} S_n) \Bigg\}. \end{aligned} \tag{3.13}$$

A numerical solution is obtained by minimizing V_{eff} with respect to R_n and S_n. In figure 3 a particular solution is displayed which is different from the usual symmetric solution with $S_n = 0$ of figure 2. Here we started with $\sigma \neq 0$ and reduced σ to 0. It is seen that we are trapped in a minimum of V_{eff} for which $S_n \neq 0$. This provides numerical proof for the existence of multiple solutions. From these numerical recurrence coefficients one can evaluate all correlation functions and free energies.

The multiple solutions found above show differences at the non-perturbative level (or double-scaling limit), and in the correlators and eigenvalue distributions. At higher orders in $1/N$ the symmetric solutions satisfy the Painleve II equations while the asymmetric solutions satisfy the modified Painleve II equation. Certain correlators are also different for these solutions (details can be found in reference [5]). The eigenvalue distributions can also be studied. There is a possibility that the multiple solutions differ due to one-eigenvalue effects; their bulk effect is the same.

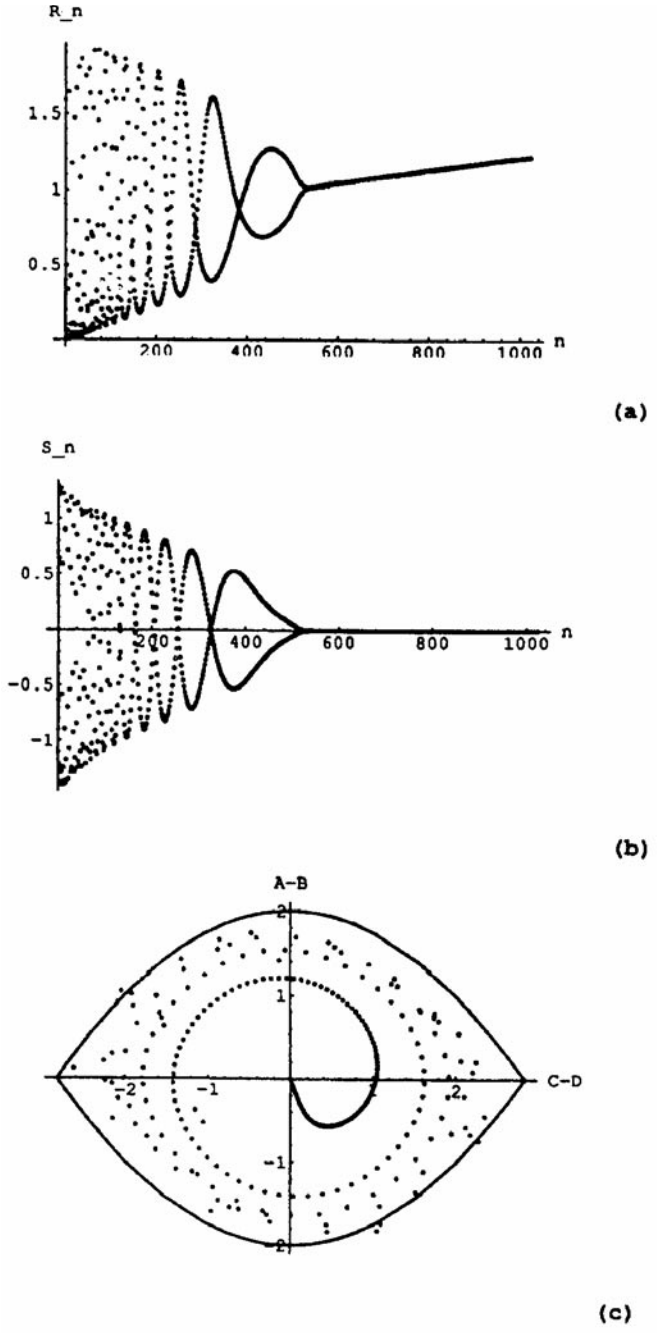

Figure 3. The recurrence coefficients (a) R_n and (b) S_n for an asymmetric solution to the symmetric double well after 100 000 minimization steps from a random start. (c) The orbit in the $A - B$ versus $C - D$ plane. The parameter values are $N = 512$, $\mu = -2$, $g = 1$ and $\sigma = 0.1$.

Furthermore, recently we studied the smoothed or long-range two-point density–density correlation function $\langle \rho(x)\rho(y)\rangle_c$ (a physical correlator in these models) in reference [6] using

the method of steepest descent. It is found that this correlator is determined up to a constant C which cannot be fixed by the method of steepest descent. For the 'symmetric solution' the constant C can be fixed; its value is $(-1)^N$ [6, 8] (this solution corresponds to first taking the limit $\sigma \to 0$ and then taking the limit $N \to \infty$). Other values for C were found earlier using the loop equation method [7] (where first the $N \to \infty$ limit is taken, followed by the $\sigma \to 0$ limit); this gives further support for the existence of multiple solutions in these models.

Another interesting property is that the correlation functions (for example the $\langle \rho(x)\rho(y)\rangle_c$ correlator) tend to two different thermodynamic limits as we send $N \to \infty$ along different sequences (odd and even). This unusual behaviour was suggested as a possible scenario in the study of systems without quenched disorder with a complex landscape in reference [3] and is present in a related spin-glass model [9].

Now I turn to the Gaussian Penner model discussed at the beginning of this section, which is closer to the random-matrix models that arise from a simple model of structural glasses. As for the M^4-model, we take the period-2 *ansatz* for the recurrence coefficients R_n and S_n (equation (3.9)). Then the recurrence relations for the Gaussian Penner model reduce to four conditions (two each for n even and n odd) but, as before, only three equations are independent:

$$\begin{aligned} &C + D = 0 \\ &A + B - CD = \frac{2x + t}{\mu} \\ &AB = \frac{x(x + t)}{\mu^2}. \end{aligned} \tag{3.14}$$

Thus again there is an infinite family of solutions labelled by one function. Let us take as an example two extreme solutions of this infinite family of multiple solutions, the symmetric solution and the maximally asymmetric solution. For the symmetric solution,

$$C = D = 0 \qquad A = \frac{x}{\mu} \qquad B = \frac{x + t}{\mu}$$

while for the maximally asymmetric solution,

$$A = B + \frac{1}{\mu}\sqrt{x(x + t)} \qquad C^2 = \frac{1}{\mu}[(2x + t) - 2\sqrt{x(x + t)}].$$

Once again, in equation (3.14) we find the same combinations as appear in the generating function of the Gaussian Penner model. The same three combinations as are fixed by equation (3.14) occur in the generating function equation (3.12). Thus in the large-N limit the eigenvalue densities and free energies are identical for the symmetric and maximally asymmetric solutions.

In the non-perturbative region the above two solutions have different free energies [5, 10, 11]. For the symmetric solution, as the exact recurrence solutions are known, the exact free energy may be found to be

$$\Gamma_{sym} = \sum_{k=1}^{N/2-1} k \log[(2k + \mu + 1)(2k + \mu - 1)]$$

where $t = -1 + \mu/N$. Expanding in powers of μ we get

$$\Gamma_{sym} = \frac{1}{4}\mu^2 \log \mu + \frac{1}{12} \log \mu + \cdots.$$

Note that the coefficient of the second term $\log \mu$ is $\chi_1 = 1/12$, which is the torus contribution. The asymmetric solution is more difficult, but in the double-scaled limit

$$R_n \approx \frac{\Gamma(1/2(N - n + \mu + 3/2))}{\Gamma(1/2(N - n + \mu + 1/2))}$$

and the double-scaled free energy is

$$\Gamma = \sum_{k=1}^{N/2-1} k \log[(2k + \mu + 1/2)(2k + \mu - 1/2)] + \cdots .$$

On expanding in powers of μ, the free energy is obtained as

$$\Gamma_{asym} = \frac{1}{4}\mu^2 \log \mu - \frac{5}{48} \log \mu + \cdots .$$

The coefficient of the second term is $5/48$. Here in the Gaussian Penner model we see that though the symmetric and asymmetric solutions give the same answer in the large-N limit, the free energies are very different at higher order. This establishes that in the Gaussian Penner model this unusual phenomenon of multiple solutions is present; they give the same free energy in the large-N limit but different ones at higher orders. Hence the random-matrix models with gaps in the eigenvalue distributions which are related to glassy models are expected to have this special property.

The questions that would then be of relevance to the structural glass models in the high-temperature phase are those of what the correct free energy is in the high-temperature phase (is it Γ_{sym} or Γ_{asym} or any of the other infinite answers corresponding to the multiple solutions), and what the correct susceptibility is etc.

4. Conclusions

Connections with the high-temperature phase of a simple model of structural glasses have been made to matrix models as discussed in references [1, 2]. These are a variant of the Penner or Penner–Kontsevich model with two cuts. A simpler model, the Gaussian Penner model, which has disconnected eigenvalue segments, is shown to have these unusual multiple solutions as well. A generic toy M^4-matrix model has also been studied in this context. First a simple reason for expecting the presence of multiple solutions is given and then the formulation is made more precise using the recurrence coefficients of the orthogonal polynomials of the M^4-model. Further, numerical evidence is given for the existence of multiple solutions in this context. In the large-N limit the free energy, generating function and density are the same. Differences between the multiple solutions are seen in the non-perturbative solutions as well as the correlators, eigenvalue distributions and free energies at higher order in $1/N$. The two-point density–density correlator tends to different limits as N goes to infinity through different sequences (odd or even). This behaviour is similar to that suggested in another model for glasses [3, 9].

Since the 'glassy' matrix models of [1, 2] also have gaps in their eigenvalue spectra, it is expected that they will also have such multiple solutions. Questions regarding what the correct free energy or susceptibility is will have to be addressed again, as now there are many answers. The ruggedness of the landscape and the number of multiple solutions need to be studied. It would be nice to be able to cast these models in the replica framework, but this remains a difficult task at this point, as the Hubbard–Stratonovich transformations which are technically needed for the Gaussian random-matrix models [12] are not available here for the simple M^4-model and the Gaussian Penner model or any other gapped random-matrix models. Numerical methods also need to be explored. This is a future goal in this problem.

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